William Robert Porter

BUDGET ALLOCATION FOR EFFICIENT FORCE STRUCTURE

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by
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APPROVAL SHEET

Title of Thesis: Budget Allocation for Efficient Force Structure

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Master of Arts, 1967

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ABSTRACT

Title of Thesis: Budget Allocation for Efficient Force Structure
William Robert Porter, Master of Arts, 1967
Thesis directed by: Clopper Almon, Jr., Ph.D.
William A. Niskanen, Ph.D.

When a military force commander makes a recommendation for the composition of his forces, he usually wants more, and more is usually more effective. A decision-maker must weigh the recommendations of several force commanders in terms of suitability of the force capabilities to national objectives, and of program costs to total defense budgets. In this thesis, the parametric solution known to economists as the expansion path is proposed as a particularly suitable form of analysis of force structure.

The expansion path displays efficient combinations of force components which have greatest military effectiveness for any selected budget, or which deliver a given effectiveness for least cost. An efficient force is defined in terms of two cost-effectiveness criteria, performance on a one-year level budget and in an N-year planning period. The duration of the planning period, N years, is found to be related in a very natural way to force composition, effectiveness, and an imputed interest rate.

Economic theory has heretofore recognized the expansion path but paid scant attention to the influence on the path of existing assets, duration of planning period, or possibility of contraction. These matters are of considerable interest in force structure analysis, and their influence on the expansion path is shown.

The expansion path is modified by the existence of inherited assets, and as the assets change, the path changes. The influence of existing assets is easily visualized through the mechanism of broken-line budget constraints. In lieu of the usual linear or smooth budget constraints, they are shown with violent changes in slope at critical input levels. The result is also pronounced in

the contraction phase of force structure changes. In particular, the paths show some irreversibility, and proportions that were efficient in some condition are not likely to be efficient after a change, even though the production function for these inputs has not changed.

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by
William Robert Porter

Thesis submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
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1967

1967
PORTER. W.

PREFACE

As an officer designated for engineering duty in the United States Navy, I have enjoyed technical duty in the study, design, and production phases of defense systems acquisition. In this time I had contact with "the other world" of engineering cost estimators, budget analysts, and financial managers. It is scarcely possible to grow older in the technical service without increasing one's involvement in the "other world." In my case, this kindled a strong interest in the academic principles involved, in lieu of, say, the volumes of administrative rules and detail, or the mechanics of data processing, which might represent some of the essential trades in "that world." Some voluntary homework in the new language of "this world" made it clear that the foundations of organized knowledge were in the social science of economics. I am thankful that the University of Maryland and the Institute for Defense Analyses offered a cooperative course in resource allocation and defense policy, or Defense Systems Analysis, if you wish. This course, sponsored by the Department of Defense, in my case, was an ideal introduction to citizenship in a new world. I am grateful that my service and the Ship Systems Command gave me a passport, and I am convinced that there should continue a growing cultural exchange program. The administrative steward of the program, Brigadier General S. F. Giffin, U.S.A.F.(Retired), is due credit for encouraging all my colleagues in the program. My greatest academic debt is due Dr. William A. Niskanen, who was simultaneously an academic director of the program, a formal teacher, and my thesis advisor at the Institute for Defense Analyses. An economist, Dr. Niskanen, with personal experience within the Department of Defense on force analysis and allocation problems, is eminently qualified to guide such as I across the bridge. His interest and engaging way make the experience most enjoyable as well as fruitful. I am grateful

to him. I am thankful that Professor Clopper Almon became my thesis supervisor at the University of Maryland, and that I could meet him in this way. My meetings with Professor Almon helped remove the deficiency that I could not also be his student. From that I would have also benefited; this would have increased my appreciation for our contact. The thesis private discussions are a separate joy from classroom development, however, and are treasured.

Now the thesis itself. In my new world I have learned that the foundations I was seeking are in the principles of resource allocation within the disciplines of economics. I enjoyed working on this exercise in the field under my two outstanding advisors. My wife, Jeanne, is the only one to whom I am more grateful and indebted.

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CHAPTER I

THE PROBLEM: TO UNDERSTAND THE EFFICIENT ALLOCATION OF BUDGET RESOURCES FOR CHANGES IN FORCE STRUCTURE.

A. Purpose.

How should a military force commander decide on his recommendation for the expansion of his force components? Should he ask for the same percentage increase in two components to keep the ratio constant? Should he refer the question to the two component commanders and base his recommendation on the most compelling argument given by one of the component commanders? Maybe he should recommend the biggest possible purchase on this year's appropriation? Should a cut be across-the-board? Is a contraction in the same proportion used for expansion?

If the recommendation is requested by a senior decision-maker, what guidance would be most useful to the force commander? What form of recommendation and substantiating argument might the decision-maker find useful in making his decision? Should he insist on the force composition which will be the cheapest to operate, or the cheapest to purchase?

We assume that both force commander and decision-maker want an efficient force, and an orderly expansion, or contraction.

What is an "efficient" force? How could they tell whether the existing force was "efficient" and whether a proposed change in force composition was "efficient?"

This thesis will shed some light on the questions. To do this we will (1) investigate the conditions defining an "efficient" expansion or contraction of a military force, and (2) describe a procedure for finding this "efficient path" among the infinite

alternatives. It turns out that the solution and procedure also illuminate the difference in the roles of the force commander and the decision-maker.

B. Analogy in Economic Theory.

The problems of the force commander and decision-maker are "how much" and "who gets it." That makes the question one of resource allocation, an important part of economic theory.

Within microeconomics the Theory of Production addresses itself to the efficient allocation of factors of production to alternative plans. We will draw on this precedent as a Theory of the Force to study the efficient combination of force components for military effectiveness.

Utility Theory, also within microeconomics, studies how consumers maximize utility, the reward of consumption, subject to budget constraints. We draw on this as an Effectiveness Theory, which studies how defense planners maximize effectiveness, the purpose of purchases and operating appropriations, subject to budget constraints.

In force analysis, the volatility of demand forces us to place more emphasis on two considerations that are less prominent in economic analysis of industrial production or utility analysis.

(1) In common with other public goods, there is no dollar value on the output, military effectiveness. To "provide for the common defense" is a specific motivation to ordain and establish a constitution and is a service not to be denied a nation seeking to "secure the blessings of liberty to ourselves and our posterity." Although citizens or divisions of the political economy may benefit unequally from protection, there is no pricing mechanism to establish the value of this service and the added cost for added customers is zero.

The absence of dollar value on effectiveness is much more fundamental than just a question of the "units" of effectiveness and the units of dollar budgets. Optimal levels, the demands, are a political judgment not priced in a market place and not amenable

to decision from the analysis of force structure alone. This deficiency of the analysis dictates that the results of a force study will be parametric, presenting "efficient" solutions for feasible levels of cost and effectiveness. Our procedures emphasize the parametric form of the results and should be valid for other analyses which have a volatile or a political demand.

(2) There is usually a large difference in the value of existing military assets in continued military use and in alternative uses. The scrap or salvage value of military hardware is generally insignificant compared to the investment cost of that hardware or its substitute. The investment cost of new force units is often very large compared to annual operating costs of existing units. The practical significance of this is that the "efficient" combination of force units will be strongly influenced by the composition of existing forces, and the analysis must take account of existing forces.

The same large difference in value in continued use and in alternate use would exist for important and expensive examples of industrial investment (refineries must refine, large rolling mills must roll) except that the demand is in the long run more stable, predictable, priced, and not political, and the industries are not monopolized. The analysis that follows would suit other economic examples in which the demand on the total industry is volatile or provided by a monopoly and there are simply no resale values in the assets.

The strong influences of existing force components on "efficiency," together with disparity in investment and operating costs, suggest caution in interpreting difference in force composition as indicating inefficient behavior. We will propose tests of efficiency. The result is that a ratio of 3 to 2, motorized rifle to armored forces, may be efficient for Country A, while 4 to 1 is efficient for Country R. In an expansion or contraction, the ratios could reverse, or change in some other confusing way, while remaining efficient for each country.

C. Lessons from a Simple Example.

Three examples will illustrate some of the characteristics of an analysis and will focus attention on some complications introduced in attempting to apply the procedure to more complicated force-analysis problems.

What analysis can we apply to all the following force structure problems? Let us say there are no existing forces, and we will introduce that complication later.

(1) The commander of strategic retaliation forces must recommend his force structure. Only two weapon systems are going to be available and reliable in the immediate future: a submarine-launched ballistic missile (SLBM), and a land-based intercontinental ballistic missile (ICBM). The SLBM has a range of 2,000 nautical miles and a 500-kiloton warhead. The ICBM has a semi-circumference range and any of 1-to 60-megaton warheads. The SLBM has very high probability of surviving any surprise attack, and can be relied upon to retaliate against their assigned targets. Due to their limited range and warhead size, the SLBM would not be assigned to all targets. The very high probability of being able to retaliate is the motto of the submarine force. The ICBM can reach and destroy all targets, but the enemy strategic force will know the ICBM locations and he can destroy many of them. The ability to destroy any target is the motto of the missile force.

(2) The ground force commander must recommend some mix of motorized infantry forces and armored forces. The motorized infantry forces are the pride of the modern counterpart of foot soldiers and horse cavalry, with well-developed tables of organization and tactics for the terrain expected. The armored forces are mechanical marvels and capable of an astounding firepower. Unfortunately, the mechanical sophistication and reliability makes this an expensive force, although the annual support for the motorized infantry force is certainly not cheap. The motto of the troops is to go anywhere and control what they hold. The motto of the armor is to go fast with firepower.

(3) The sea frontier commanders must recommend composition

of their anti-SLBM forces, consisting of destroy-helicopter teams (DH), and attack submarines (AS). The DH can operate at the submarine passages and on the continental shelf, as well as on the deep ocean. For technical reasons, their detection and fire control apparatus is not as effective as that on an AS. The AS can intercept the deploying SLBM submarines at passages and in the deeps, but it has difficulty near the surface or on the continental shelf.

It would be folly for any of the force commanders to recommend only one component for his force. One component alone can not achieve his objective. What objective? Let's say,

(1) Percent enemy industrial capacity destroyed after his foolish first surprise attack,

(2) Square miles per unit time gained and controlled from a withdrawing force,

(3) Percent own population surviving a rather fast build up of tension, leading to a crisis, and an enemy strike by SLBM.

The techniques of operations analysis and perhaps war-gaming as form of simulation should give approximate numerical measure of these objectives, given one pair of components. A repeat with another pair should give another estimate. In principle, an effectiveness function of the components, $\varphi = \varphi(x,y)$, could be developed. As a matter of fact, it is possible to proceed even if numerical (cardinal) evaluation of effectiveness is too difficult, if someone will reveal a preference and declare combinations which are equivalent. Since the inputs are partial substitutes, we accept the existence of equal-effectiveness combinations. Readers trained in economics will recognize the parallel to indifference maps and utility functions, or to production functions. More will be said about effectiveness functions in Chapter II.

The commanders naturally want "more." Under most conditions, "more" is also more effective. There is, however, the problem of the budget. How much will it be, and how will it be divided? A simple but immutable conservation law is that the budget will be equal to the sum of its parts. In this case, if the component

average costs are c and d for numbers of components x and y, then $B = cx + dy$ is the budget constraint. More will be said about budget constraints in Chapter II.

The force commander can state his problem in two ways:

(1) Maximize $\varphi = \varphi(x,y)$,
Subject to $B = cx + dy$,

or (2) Minimize $B = cx + dy$,
Subject to $\varphi = \varphi(x,y)$.

It is well known that, given convex isoquants of the effectiveness function $\varphi(x,y)$, these two problems are readily solved, in principle, by existing techniques, and that the two problems are logically identical. The hesitant reader may refer to Henderson and Quandt [1958], Section 3-2. The important point here is the interpretation of the analysis and its structure.

The force commander can not give a unique recommendation, even when he knows all about $\varphi(x,y)$ and costs (c,d), without specific instructions on B (Problem 1), or φ (Problem 2). The budget information will require coordination with the comptroller, or the effectiveness level φ will require coordination with national objectives and other force strengths.

The force commander can give on his own authority and analysis a display of consistent force structures (x,y) that are efficient, and consistent with effectiveness levels φ, and budget levels B. This display of consistent $[E(x,y);\varphi;B]$ we call a parametric solution to the force structure problem. The set of efficient force components $E(x,y)$ define the (x,y) decision given either the effectiveness φ, or the budget B.

Of the three,

efficient (force structure components),
effectiveness φ ,
budget constraint B ,

only one is independent.

The problem is to

(1) Define and proceed to find $E(x,y)$, and,
(2) specify the relation among $[E(x,y);\varphi;B]$.

This is done by finding the parametric solution that readers trained in economics will recognize as the <u>expansion path</u>.

The expansion path solution for the simple example problem is shown in Figure 1, if we substitute the names of force components and effectiveness measures for x, y, φ. The efficient forces $E(x,y)$ are tangent points of the budget-constraint and effectiveness curves. The expansion path shows consistent points $[E(x,y);\varphi;B]$.

Microeconomic theory recognizes the expansion path but has little more to say about it, not showing much concern for the influence of high-cost, no-resale assets, or the possibility of contraction. Complications to the expansion path will be developed, but we are still discussing the structure of the analysis.

D. Alternate Analysis of the Example.

Now we know that the expansion path solution gives force combinations that have the maximum effectiveness for any given budget level, or cost the least for a given effectiveness. We can deduce from this solution various characteristics. If we call the path-forces $E(x,y)$ efficient, then these deduced characteristics help to define efficiency. We name a few.

(1) For an efficient force, an additional dollar spent on one component will buy no more effectiveness than that dollar would buy if spent on the other force.

(2) The maximum increase in effectiveness for an additional dollar will be gained by buying the force components in the ratio shown by the path.

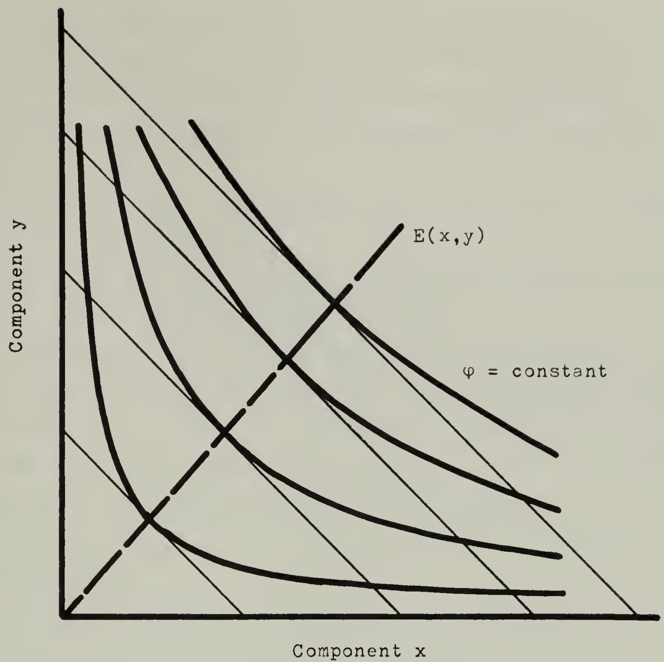
(3) The ratio of the gain in effectiveness for a gain in each component is equal to the ratio of the costs.

(4) The above are true only on the expansion path.

These are interpretations of the result which readers in economics recognize as the special marginal features of efficient solutions. Our point here is that such a collection illustrates the second facet of analysis for optimal choice.

(1) Know a method for finding the choice, and,

(2) know the characteristics of the choice.



Effectiveness:	$\varphi(x,y)$	$\varphi = x^b y^a$
Budget:	$B(x,y)$	$B = cx + dy$
Expansion Path:	$E(x,y)$	$y = \frac{a}{b}\frac{c}{d}x$

Efficient Parametric Solution: $[E(x,y);\varphi_o;B_o]$

FIGURE 1

The Efficient Path for Force Structure Expansion

Here we knew a method (Lagrange multiplier or vanishing Jacobian; the former is common in the economic literature), and then interpreted the partial derivatives that define the maximization as being characteristics of efficiency. The origins of the two parts of analysis are different.

(1) The method for maximization is mathematics or symbolic logic (graphics).

(2) The verbal statements of efficient characteristics have economic content and origin.

In Chapter III we state characteristics of efficiency first, and then employ suitable methods. The result will recover all the nice features of the solution to the simple example, and add some.

The alternate approach still leaves the level of budget and effectiveness to be determined by the decision-maker.

E. Information Output from the Analysis as Input to Higher Order Analysis.

The first result of the procedure used in the example (and later) is the expansion path $E(x,y)$, an implicit function of all force components that has marvelous properties. The function $E(x,y)$ forces the functions $\varphi(x,y)$ and $B(x,y)$, which were independent, to be dependent in a way that is very significant to both force commander and decision-maker. The expansion path $E(x,y)$ shows the particular force structures (x,y) which produce the greatest effectiveness for a given budget, or provide a given effectiveness level at minimum cost. The complete parametric solution $[E(x,y);\varphi;B]$ displays efficient for combinations, effectiveness levels, and budget totals.

The display of consistent effectiveness, budgets, and compositions is the essence of a cost-effectiveness study, or cost-benefit analysis, with some particularly favorable characteristics. It has not been necessary to assign dollar values to effectiveness. In principle, effectiveness does not require a numerical measure. No questionable criteria such as benefit-to-cost ratio, or difference, has been invoked. The complete result does display the feasible frontier of output possibilities and the associated cost.

Formally, the parametric "efficient expansion path" contains all the information of a supply function, so that the consequence of various demand functions and levels of demand may be seen.

The total program cost and effectiveness are given by the pair (φ, B) from the relation $[E(x,y);\varphi;B]$ and the marginal cost is the total derivative

$$\frac{dB}{d\varphi} = \frac{\partial B/\partial x}{\partial\varphi/\partial x} = \frac{\partial B/\partial y}{\partial\varphi/\partial y} \quad , \quad \text{given} \quad E(x,y) \quad .$$

Another, more subtle, advantage of the parametric solution $[E(x,y);\varphi;B]$ is that it tends to force the commander and the decision-maker to concentrate on their own roles. The commander can best concentrate on operations analysis and brilliant leadership to get highest effectiveness from any force, and the decision-maker can do his most valuable service outside the operations room, making firm policy guidance among different program objectives and total national budgets.

F. Introduction of Complications.

The next three chapters introduce complications into the basic problem and generalize the procedure for finding the efficient solution and defining its characteristics. The result will still be a parametric solution, a display of $[E(x,y);\varphi;B]$, which is an expansion path in a broad sense.

The path is modified by considering an existing force. The existing force may or may not be on the path. The solution is also modified by the exhaustion of one or more alternate force components. Either existing forces, or exhaustion of alternatives, or variable cost fields, and the changes in the efficient path as the force changes, make it unlikely that a path is reversible, or the same for two different defense departments.

The following analysis helps to illuminate the alternatives and consequences in decisions to modify existing inefficient forces. For good reasons that will be seen, this problem is not fully answered here.

Before going on with efficiency and solutions, we will discuss in Chapter II the general characteristics of some new budget constraints and effectiveness functions.

CHAPTER II

SOME PRELIMINARIES ON COST AND EFFECTIVENESS FUNCTIONS.

A. Remarks.

This chapter is divided into two parts. Part I, Sections B-C, introduces cost functions and budget constraints. Part II, Sections D-G, gives a general classification of the effectiveness functions and their properties. No attempt is made in this chapter to associate production and cost. That will be done in following chapters.

PART I. COST FUNCTIONS AND BUDGET CONSTRAINTS.

Cost functions and budget constraints are now made more general than those used in the simple example of Chapter I. In that example, the marginal cost of each force component was constant for all force levels. We introduce a major generalization in admitting that marginal costs may be discontinuous at discrete levels of input. A simple example is to consider operating costs for existing forces (x_o, y_o) and to add investment costs for new procurement.

Example:

Operating costs are	O_x , O_y	per unit for all x, y .
Investment costs are	I_x	per unit in excess of x_o .
	I_y	per unit in excess of y_o .

Although the cost function in this example is discontinuous at the critical levels x_o and y_o, the budget total is continuous. The lines of constant budget (budget constraints) on the x-y plane are therefore piecewise-linear, with changes in slope at critical force levels. The reader may wish to glance ahead briefly at Figures 2 and 3 which illustrate broken-line budget constraints and the two different ways that we will consider budget constraints on force structure. The two different views are developed in the following way.

(1) Vary the duration of the N-year planning period, but keep the average annual expenditure constant and equal to the current operating budget B_o implied by the existing forces. This is the N-year level-budget constraint and is discussed in Section B and illustrated in Figure 2. Typical time-streams of expenditures under this constraint are shown for N=1 in Figure 4 and for N-years in Figure 6.

(2) Vary the average annual expenditure, with the N-year planning period fixed. In actual analyses, N is often ten years. To vary the average expenditure over the N-year period is simply to adjust the total budget level B.

The budget level B must be taken as a parameter, because the analyst will not likely have authority to set the B-level at a particular value. If he has guidance from a senior, he is still prudent to consider B as a parameter (centered on the edict) in order to illuminate the problem for a rebuttal, unsuspected economies, or in preparation for a budget change. This B is the parameter B in the complete display $[E(x,y);\varphi;B]$ giving efficient, consistent, force combinations, effectiveness and B-levels. Parametric B budget-level constraints are shown in Figure 3 and discussed in Section C.

B. N-Year Level-Budget Constraints.

The reasons for introducing this budget constraint are:

(1) It will help to define "efficient" force combinations in Chapter III.

(2) It will be used to investigate the meaning of the interest rate with respect to modification of force structure in Chapter III.

(3) It helps to introduce the concept of feasible component combinations; an efficient force must be feasible within the parametric budget level.

(4) It is the appropriate place to introduce the program planning period, N-years.

The level budget in this constraint is a fixed total budget over the N-year program planning period. The force strengths at which costs abruptly change are identified, such as (x_o, y_o), and might indicate that a dominant suboptimization switches alternatives in a composite force, or be the size of existing forces. The budget equal to the N-year total implied by operating the existing force (x_o, y_o) over the N-years is

$$NB_o = N(O_x x_o + O_y y_o) \quad ,$$

and the average yearly expenditure is B_o, the current force operating cost.

Any other combination which could be operated on the same budget NB_o is on the limit line

$$NO_x x + NO_y y = NB_o$$

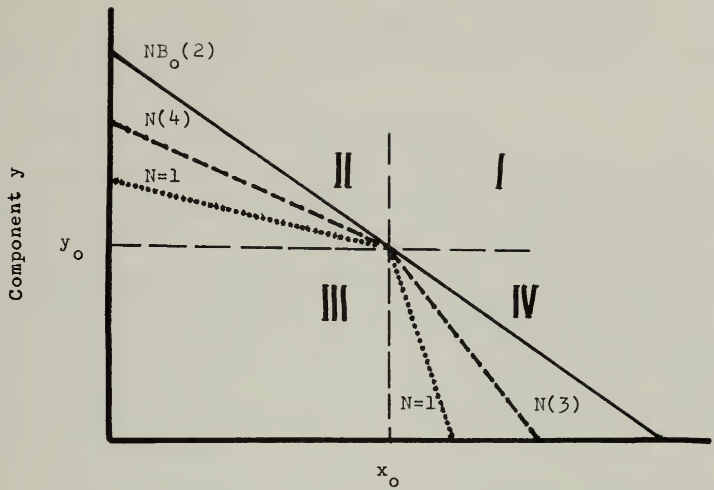
shown in Figure 2. No force combination to the upper right of this line could be operated, or purchased and operated, under this level-budget constraint.

Any other force combination (x,y) is feasible which can be purchased and operated with the same N-year total program cost, NB_o. It is not feasible to purchase both more x and more y (quadrant I), but more x (and less y, quadrant IV), or more y (and less x, quadrant II) are feasible. How much "more" depends on the duration N of the program planning period.

The corresponding N-year level-budget constraints are shown in Figure 2. These budget frontiers represent combinations of components (x,y) which are feasible over an N-year planning period at a total program budget equal to N-times the annual operating cost for the existing force.

The broken lines all pass through (x_o, y_o). As the planning period N increases, the N-period segments in quadrants II and IV approach the limit line with slope O_x/O_y, the ratio of operating costs.

$$\frac{NO_x}{I_y + NO_y} \rightarrow \frac{I_x + NO_y}{NO_y} \rightarrow \frac{NO_x}{NO_y} = \frac{O_x}{O_y} \quad , \quad \text{for } N \rightarrow \infty \ .$$



Component x

1. Definition of level budget.

$$NB_o = N(O_x x_o + O_y y_o)$$

2. Limit line for feasible force compositions.

$$NO_x x + NO_y y = NB_o$$

3. Feasible forces in quadrant IV.

$$(I_x + NO_x)x + NO_y y = (NB_o + I_x x_o)$$

4. Feasible forces in quadrant II.

$$NO_x x + (I_y + NO_y)y = (NB_o + I_y y_o)$$

FIGURE 2

N-Year Level-Budget Constraints

The focus of the N-year level-budget constraint is strongly dependent on the existing forces, and this field of broken lines shifts as the force composition changes. The level budget NB_o is a budget concept and it is neither the "actual" budget, nor the parametric budget level B.

C. Parametric B Budget-Level Constraints.

The parameter B represents the total budget in dollars. The budget constraint for most B-levels is a broken line, piece-wise-linear. The slope of a budget constraint changes at the force level corresponding to a change in marginal cost. The critical force levels with cost changes thus form boundaries of regions where budget constraints have different slopes. The entire space of possible force combinations is divided into two by each different critical level, so there are four quadrants around the intersection (x_o, y_o) in two-force problems.

Figure 3 illustrates the case of a critical level at (x_o, y_o). At greater x, or greater y, or greater x and y, the marginal costs have increased, changing the slope of the budget constraints.

Two examples follow.

(1) Existing forces $(x_o y_o)$. Number the quadrants in the usual way, counterclockwise from the upper right. The costs and budget constraints for a given N-year planning period are as follows.

Quadrant	Conditions	Budget Constraint
III	$x < x_o$ $y < y_o$	$NO_x x + NO_y y = B$
IV	$x > x_o$ $y < y_o$	$(I_x + NO_x)x + NO_y y = B + I_x x_o$
I	$x > x_o$ $y > y_o$	$(I_x + NO_x)x + (I_y + NO_y)y = B + I_x x_o + I_y y_o$
II	$x < x_o$ $y > y_o$	$NO_x x + (I_y + NO_y)y = B + I_y y_o$

The slopes of the budget constraints in each quadrant are related. Using the quadrant number to indicate the magnitude of the slope,

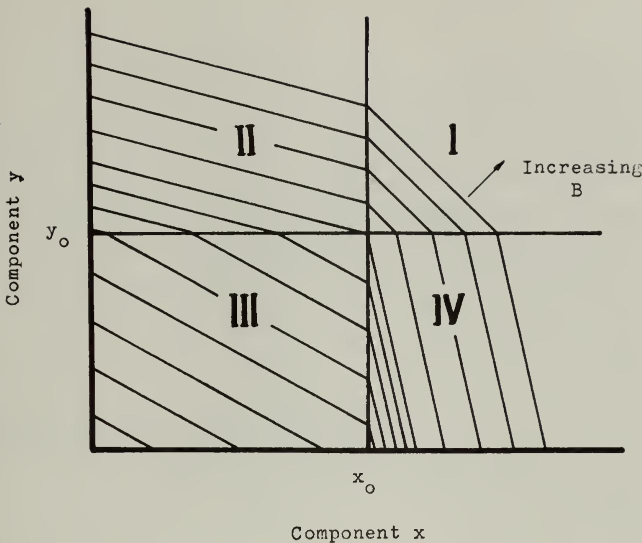


FIGURE 3

Parametric B Budget-Level Constraints

$$\text{II} < \text{III} < \text{IV}$$
$$\text{II} < \text{I} < \text{IV}$$
$$\text{I} \lesseqgtr \text{III} \quad .$$

(2) Limited Resources. Let x_1 , x_2 be substitutes for the force x with costs c_1 , c_2 respectively, which reflect any small differences in performance. Assume that x_1 is available only to level x_o. We expect that at small force levels, the cheapest x will be purchased. Although it is not necessary, the point is illustrated if x_1, which exhausts at x_o, is cheaper. If this is not the case, who cares about x_1 ? The x-force budget is

$$B(x) = \begin{cases} \text{Min}\ [c_1x_1 \ , \ c_2x_2 \ ; \ \varphi] & \text{if} \quad x_1 < x_o \ , \\ c_2x_2 \ , & \text{otherwise} \quad . \end{cases}$$

Similarly for the y-force component,

$$B(y) = \begin{cases} \text{Min}\ [k_1y_1 \ , \ k_2y_2 \ ; \ \varphi] & \text{if} \quad y_1 < y_o \ , \\ k_2y_2 \ , & \text{otherwise} \quad . \end{cases}$$

The parametric budget B is

$$B = B(x) + B(y) \ .$$

The two types of budget constraints will be used in the following chapters. The N-year level budget is a conceptual budget of variable duration N but constant average expenditure B_o which shows feasible force combinations with constant N-year cost, NB_o. The parametric B budget level shows the feasible force combinations at a total budget of B dollars in a given N-year planning period. In the first constraint, the program planning is changed and the average expenditure is fixed. In the second constraint, the planning period is fixed, but the total budget is varied over all levels that might be of interest to the analyst and decision-maker.

PART II. EFFECTIVENESS FUNCTIONS.

Effectiveness functions that we consider must have isoquants convex to the origin. Functions with linear isoquants are included as a limiting case. The more general convexity condition is consistent with the principles of diminishing returns: an addition of one

force component, with other components fixed, increases effectiveness at a diminishing rate. Convexity is an expression of the increasing rate of substitution: more and more of one force component is required to keep constant effectiveness as it substitutes for each unit of another force that is withdrawn. The essence of a study in the efficiency of force compositions is a study of the elasticities of substitution, and not just the amount of the effectiveness. Because of the importance of the elasticities, we introduce definitions of these concepts and characterize our effectiveness functions using them. We will show the wide classes of actual functions that have constant elasticities. An estimate of the relative changes in effectiveness for substitution of forces, at about the reasonable force ratios, must be easier than cut-and-try function-hunting. At least if one of several simple assumptions can be made about an elasticity, we can be encouraged immediately to know that there is a simple parametric, expansion path, solution.

The reader familiar in a general way with indifference maps of utility or convex production isoquants may prefer to skip ahead to the next chapter rather than continue now with preliminaries on effectiveness functions.

D. Substitution of Force Components: Elasticities of Substitution.

We now define three elasticities of substitution which apply to any force components, taken two at a time, which are components of a single force with a common objective which is ordered by an effectiveness function φ. Our assumption on φ is that it be differentiable, and our economic and military realists will expect it to have isoquants convex to the origin. The limiting case has linear isoquants which are convex but not "strictly convex." The reason we define the elasticities is that it is sufficient to make a proper assumption about any one of them to know that a solution $[E(\underline{x});\varphi;B]$ exists. It is not necessary to make any assumption, because more complicated path solutions for more complicated force structure problems can exist. The many, many effectiveness functions admitted by our simple approach should make our point that parametric solutions to structure problems are good.

Definition 1: The rate of substitution[1] of component y for component x in the force with effectiveness $\varphi(x,y)$ is

$$\varepsilon_{yx} = -\frac{dy}{dx} = \frac{\frac{\partial\varphi}{\partial x}}{\frac{\partial\varphi}{\partial y}} .$$

This measure expresses the ratio of the increments of forces, one increasing and one decreasing in the normal case, which will maintain constant effectiveness. It is a positive number.

If the ε_{yx} is a constant for two components, then those components are substitutes in any quantity in the ratio ε_{yx}, as far as effectiveness is concerned. This is a way of expressing the limiting case of linear isoquants. The budget constraint will normally signal the selection of the component with the highest marginal effectiveness per dollar and ignore the other, completing a suboptimization on this pair.

For general convex isoquants, the rate of substitution increases as the first-named component increases. A point where ε_{yx} is equal to the ratio of marginal costs is a point on the expansion path. If ε_{yx} is not equal to the ratio of marginal costs, move in the direction to make it so.

Definition 2: The partial elasticity of effectiveness of component x is

$$\pi_x = \frac{\partial(\log \varphi)}{\partial(\log x)} = \frac{x}{\varphi}\frac{\partial\varphi}{\partial x} = \frac{\left(\frac{\partial\varphi}{\partial x}\right)}{\left(\frac{\varphi}{x}\right)} \cong \frac{\left(\frac{\Delta\varphi}{\varphi}\right)}{\left(\frac{\Delta x}{x}\right)} .$$

This measure expresses the ratio of proportional changes in effectiveness and component x, or the ratio of marginal to average production of effectiveness φ for component x. For example, a five percent increase in troops that produces a ten-percent increase in effectiveness indicates that, with other components fixed, the partial elasticity of effectiveness with respect to troops is two, and that the ratio of marginal to average effectiveness is two. Other things

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1 Sometimes, redundantly, called a marginal rate.

being equal, $\pi > 1$ means that the added troops are improving the average effectiveness. This may be good, if the costs are favorable.

Any of the following are characteristics of effectiveness functions with piecewise-linear expansion path solutions.

(1) The π is constant for each component separately.

(2) The sum of the π is constant.

(3) The ratio π_x/π_y for pairs of components is constant.

<u>Definition 3</u>: The <u>total elasticity of substitution</u> of force components (x,y) for constant effectiveness φ is

$$\eta = \frac{\frac{x}{y}\, d(\frac{y}{x})}{\frac{\varphi_x}{\varphi_y} d(\frac{\varphi_y}{\varphi_x})} \quad .$$

This measure is the ratio of the proportional change in component quotients, $d(\frac{x}{y})/(\frac{x}{y})$, to the proportional change in the rate of substitution,

$$d(\frac{\varphi_x}{\varphi_y})/(\frac{\varphi_x}{\varphi_y}) = d\varepsilon_{yx}/\varepsilon_{yx} \quad .$$

This elasticity is symmetrical so that $\eta_{xy} = \eta_{yx}$. The extreme values of $\eta = 0, \infty$ are interpreted as follows.

(1) If the components (x,y) must be used in fixed proportion, then $y=kx$ and $d(y/x)$ is zero. In general the denominator of η is not zero, so $d(y/x) = 0$ means $\eta=0$. That is, $\eta=0$ signifies that the inputs are not at all substitutable, and for a given effectiveness φ, they will be used in proportion $y=kx$ only. The expansion path for these two components must follow this proportion.

(2) If the ratio of marginal products (φ_x, φ_y) is constant, the marginal rate of substitution is constant, then $d(\varphi_y/\varphi_x) = 0$ and the denominator is zero. This infinite value for η corresponds to a constant value of the rate of substitution ε, and the same limiting case of linear isoquants is indicated. The component forces are indefinitely substitutable for constant effectiveness at a fixed rate.

For values of η between 0 (non-substitutes used together in constant ratio) and ∞ (perfect substitutes with a constant ratio of substitution), the inputs are more or less substitutable by this measure. As η increases, the substitution of one input for the other, at constant effectiveness, is "more" feasible.

We will use these three elasticities, ε, π, η, as characteristics of our effectiveness functions.

E. Production Functions for Force Components that are Substitutes: Linear Isoquants.

There are four reasons for separating out this special case of effectiveness functions.

(1) The case is readily identified if by physical reasons it is known that two of the force components are substitutes in some fixed ratio. It may not be obvious that in some employment they are.

(2) The effectiveness function for either component alone, or for the joint action of the components on the objective, may be nonlinear and not obviously a case for treatment by linear isoquants.

(3) The techniques of linear programming will apply.

(4) The expansion path solution will be at least piecewise linear. This may occur with more complex functions, as we shall see.

Example. Assume that the effectiveness φ_1 of a large number x of missiles of Type 1 operating alone against a large target system is

$$\varphi_1 = ax^b$$

and similarly, $\varphi_2 = cy^b$

for Type 2 missiles against the same system, where a, b, c, are positive constants.

The joint effectiveness function φ for both missiles operating together or the same target system,

$$\varphi = \left(a^{\frac{1}{b}}x + c^{\frac{1}{b}}y\right)^b ,$$

may be verified by observing that an effectiveness level may be reached by either type missile and the constant exchange ratio is

$$a^{\frac{1}{b}}/c^{\frac{1}{b}} = a_o/c_o$$

units of y for each unit of x. This effectiveness function and the functions for each component are nonlinear and show diminishing returns, as $0 < b < 1$. By inspection of the parenthetical term in φ, however, linear isoquants are

$$a_o x + c_o y = \text{constant} .$$

The (negative) slope of the effectiveness isoquants is the marginal rate of substitution ε_{yx} and is the exchange ratio.

In this example we were fortunate in being able to "see" the isoquants and exchange ratio inside the parenthesis of the effectiveness function. Suppose we didn't see that? We could use any of the following.

(1) The rate of substitution ε_{yx} is constant.

(2) The total elasticity of substitution η is infinite.

(3) The following theorem.

<u>Theorem 1</u>. If two force components are perfectly substitutable at a constant rate, the joint effectiveness function is of the form, generally nonlinear,

$$\varphi = F[(1-\delta)\, x + \delta y]$$

where F is any monotonic differentiable function. The isoquants are linear,

$$(1-\delta)x + \delta y = \text{constant} .$$

The positive constant δ is equal to $1/(1+\varepsilon)$, where ε is the rate of substitution ε_{yx}, the number of y units equivalent to one x unit for constant effectiveness.[1]

It is important to note that the fact that the inputs are substitutes may be known on physical grounds without knowing the

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1 This theorem is a special case of the second theorem, following, with the exponent e equal to one.

specific value of the exchange ratio. We apply this approach to the example.

The missiles Type 1 and Type 2 have different warheads, accuracy, penetration probability, reliability, and so forth, but we assume in this example that either missile could, in sufficient number, destroy all targets. Therefore, on physical grounds, the missiles are substitutes. We cannot calculate the exchange ratio on the basis of physical characteristics alone, because we need to know about the target system to study ε_{yx}. We can invoke Theorem 1. The joint effectiveness function must be of the form

$$\varphi = F[(1-\delta)x + \delta y] \quad .$$

We select some likely monotonic function F which displays diminishing returns. It will be a concave function with linear isoquants, convex to the origin. A simple function which would approximate targeting data, with γ less than one, is

$$F(X) = X^{\gamma} \quad ,$$

$$\varphi = [(1-\delta)x + \delta y]^{\gamma} \quad .$$

This would be a reasonable exploration since it only requires plotting, on log-log coordinates, some typical targeting results. Estimates of the parameters then follow . For our purposes we only want to stress that whatever reasonable function is finally found for F to form the effectiveness function, the isoquants will be linear,

$$(1-\delta)x + \delta y = \text{constant} \quad .$$

F. Effectiveness Functions for Imperfect Substitutes: Convex Isoquants.

Not all force components are substitutes, as required in Section C above. At the other extreme, if the components are used in a rigid combination, there is no problem in deciding what force composition is efficient. The problem now is to open up the middle ground where each pair are imperfect substitutes; that is, the exchange rate changes, and the total elasticity of substitution η is between zero and infinity.

We must compromise between generality and specificity. In

principle, any effectiveness function that has convex isoquants is admissible; there probably will be an efficient composition. Unfortunately, computational techniques for finding efficient combinations have not yet been developed as a practical matter for such gross generality. Perhaps one could insist that in principle the multiplier technique of Lagrange works for any number of variables and that with sufficient insight into the problem the "corner" solutions are detected. Perhaps the large-machine user will also argue that the generalized Lagrange technique due to H. Everett [1963] is sufficiently accurate and timely, on a large machine, for any practical purpose. For our purposes we can admit two very broad classes of effectiveness functions for which computational procedures are known.

Class 1. Effectiveness functions formed by the sum of a linear function and a concave quadratic form,

$$\varphi = k'\underline{x} + \underline{x}' A\underline{x} \quad .$$

Here x is column vector of force components; k' is a row of constants; and A is a negative definite or semidefinite matrix which can be taken as symmetric. For two components, the general form is

$$\varphi = ax + by + cx^2 + dxy + ey^2 \quad ,$$

where the initial letters are constants restricted only to assure convex isoquants. If A is all zero, φ reduced to the linear form of the preceding section. The constant vector k may be null.

Class 2. Effectiveness functions which have constant total elasticity of substitution. This class includes such forms as

$$\varphi = x^{0.4}y^{0.8}$$

used in Chapter I, or

$$\varphi = \left(\frac{L_A}{L_B}\right)^a \left(\frac{M_A}{M_B}\right)^b \left(\frac{C_A}{C_B}\right)^c \quad ,$$

Where Force-A components L,M,C oppose corresponding Force-B components and the exponents are positive constants.

<u>Theorem 2</u>. An effectiveness function with constant total elasticity of substitution η will be of the general form

$$\varphi = F[([1-\delta]x^{1/e} + \delta y^{1/e})^{e}] \quad , \quad \text{for} \quad \eta \neq 1 \quad ,$$

$$= F(x^{1-\delta}y^{\delta}) \quad , \quad \text{for} \quad \eta = 1 \quad .$$

Here F is an arbitrary monotonic differentiable function, $e = \eta/(\eta-1)$, and δ is a positive real constant. There is a restriction $e \geqslant 1$ for convexity. For the example $\varphi = x^{0.4}y^{0.8}$, one finds $\eta=1$, $\delta=2/3$, and

$$\varphi = F(X) = X^{1.2} \quad .$$

The theorem follows directly from integration of the η expression, as in Paroush [1964] and Yasui [1965]. We restrict the functions F to those with convex isoquants to satisfy the principle of diminishing rates of substitution. The principle is consistent with Theory of Production and reasonable for military forces composed of imperfect substitutes. Of course, such a liberal description admits many, many production functions. The interested reader may wish to refer to the general treatment of suitable convex functions in many dimensions in Herman Wold [1953], especially Chapter 4, <u>Individual Preference Fields</u>. This second class of effectiveness functions is introduced particularly because it leads to simple expansion path solutions for the purposes of exposition in this investigation. The reason the solutions are particularly straightforward is that Theorem 3 applies.

<u>Theorem 3</u>. Let φ be an effectiveness function of Class 2; then φ is a homothetic function and the ratio of partial elasticities of effectiveness is constant along a ray from the origin.

<u>Example</u>: For the case $\varphi = x^{0.4}y^{0.8}$,

$$\pi_x = 0.4x/\varphi$$

$$\pi_y = 0.8y/\varphi$$

$$\tau = \frac{\pi_y}{\pi_x} = \frac{2y}{x} \quad .$$

Along any ray from the origin, $y = kx$, $\tau = 2k$, constant.

These rays will be used when we find expansion path solutions in Chapter IV.

G. Decomposition of Functions.

A force of three or more components employed for a common objective may have a decomposable effectiveness function. The decomposition of the function will lead to partial functions of the two types above.

<u>Example</u>:

$$\varphi = x^a(by+cz)^d \quad .$$

In this example, the parenthetical force components are substitutes. A suboptimization is completed on these components alone to find [E(y,z); w=by+cz; B(w)], and then

$$\varphi = x^a w^d$$

is an effectiveness function of the second kind, with convex isoquants.

CHAPTER III

JUDGING "EFFICIENCY" OF AN EXISTING FORCE STRUCTURE.

A. Remarks.

The simple introductory example developed an expansion path from zero levels of input. Inherited assets, or an existing force level, may not be a combination of forces on this path. Existence of a force alters the path, and even so, the combination may not be "efficient." How would one judge the "efficiency" of an existing combination? Lacking a "Present Value of Income" or "Internal Rate of Return," we invoke the criteria presented in this chapter.

These criteria relate to costs and budget constraints as well as effectiveness; thus they are cost-effectiveness criteria. It is worth note that these criteria do not relate to what unit commanders are apt to mean by "military efficiency." It is assumed that leadership, operations analysis, manning levels, training, and consummable supplies have all assured that the effectiveness φ is already at the highest possible level for any given force composition. The commanders can maximize military efficiency, but what about costs?

B. Performance on One-Year Level Budget.

It seems obvious that a force should be as effective as possible on a one-year level budget, but it isn't necessarily true that an existing force satisfies this test. Certainly, any new combination should satisfy such a criterion. There are important concepts associated with such a simple test so we state the

<u>Criterion 1</u>: Maximum Effectiveness on a One-Year Level Budget.

This is an example of an N-year level-budget constraint with N equal to one. The value of the one-year level budget is the scheduled operating budget for the existing force,

$$B_o = O_x x_o + O_y y_o \quad .$$

If the existing force fails this test, an increase in effectiveness is possible at no added one-year cost, as illustrated in Figure 4a.

The operating budget for any force on an N-year budget constraint is less than the operating budget for the existing force. A shift to another combination on the one-year line requires only a one-year constant budget, and the budget may be reduced the second year because the operating costs are lower.

The time-stream of annual budgets and effectiveness by an immediate shift from (x_o, y_o) to (x_1, y_1) would be as shown in Figure 4b, with succeeding budgets maintaining the increased effectiveness. This is not stated as an optimal policy, but to introduce the concepts.

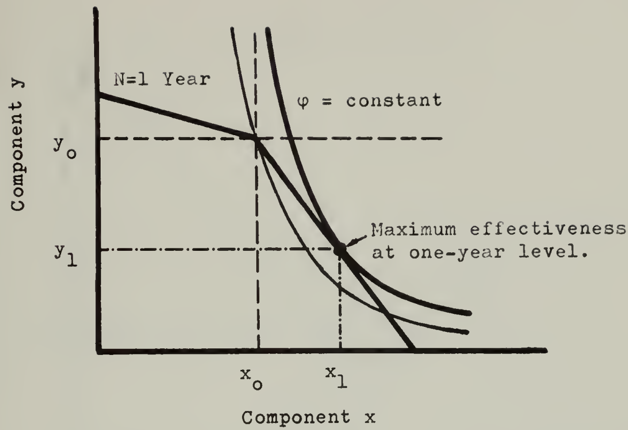
The two-year program possibilities implied by an existing force failing criterion 1 may be illustrated on a two-year budget plane as shown in Figure 5. A constant one-year budget is a vertical line. A constant two-year total budget may be divided in any proportion on the downward-sloping 45-degree budget line. The vertical and 45-degree budget lines intersect at the existing (x_o, y_o) force. The constant-effectiveness contours are for feasible (x,y) force compositions which result from procurement and operation in the first year and operation in the second year.

To purchase additional x-units and retire y-units at constant budget is to move down the one-year budget constraint in the x-y plane of Figure 4a. In Figure 5, this is to move down the vertical line of constant first-year budget.

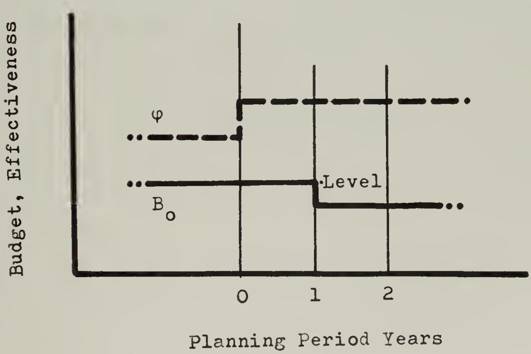
It is emphasized that the purpose here is to illustrate the concepts associated with criterion 1, maximum effectiveness on a one-year level budget, and not to argue any "optimal policy." We do define an inefficient force as a force combination, existing or conjectural, which fails criterion 1. It turns out that inefficient forces are outside a convex cone containing all possible efficient combinations.

C. Relation of Planning Period and Discount Rate.

If the existing force is "efficient" on criterion 1, we will test it for maximum effectiveness on an N-year level budget of the



4(a)



4(b)

FIGURE 4

Maximum Effectiveness on a One-Year Level Budget

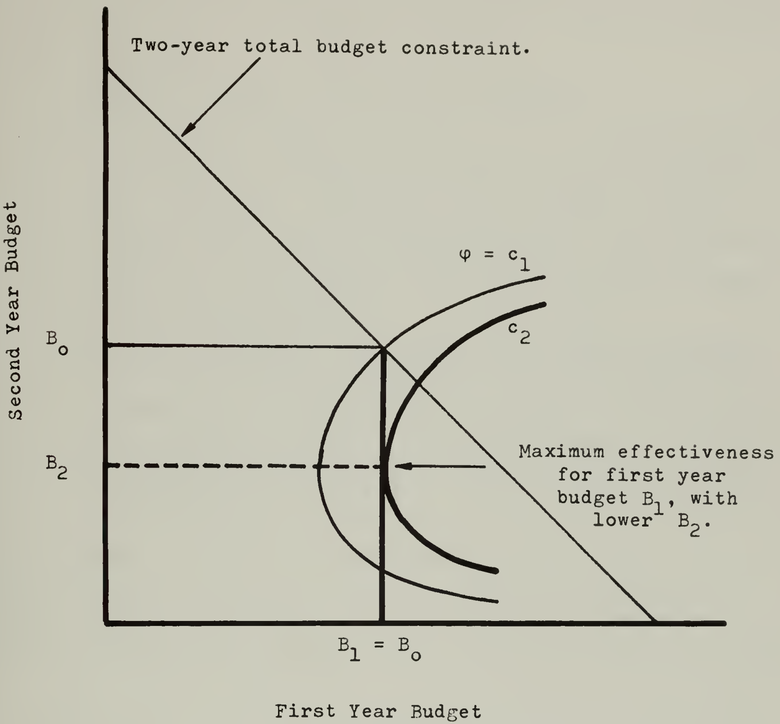


FIGURE 5

Distribution of the Budget in Two Years

same total value as the implied operating budget, NB_o. Of course an inefficient force (which failed criterion 1) cannot be efficient on any $N > 1$.

The reasons we make this test are important.

(1) This test relates the length of the program planning period, N, to the effectiveness of a force judged to be efficient. After this test, it should be clear that force structure, planning periods, and effectiveness are not separable entities in a context of cost-effective efficiency.

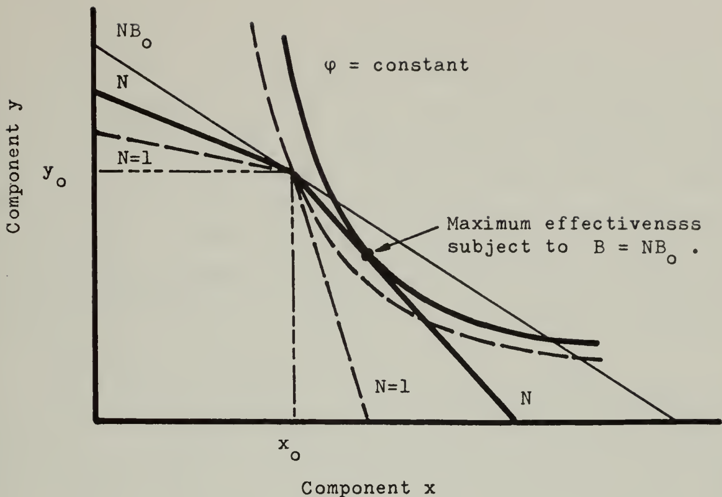
(2) This test relates the length of the program planning period N to an interest rate r in a natural way. Conversely, the effect on the analysis of an externally-established interest rate can be understood. If the exogenous interest rate is rigid, it joins the inseparable determinants of the result.

We want to make two interpretations of this test, which is easily visualized from Figure 6a. We illustrate a case with an existing force that satisfies criterion 1, but the existing (x_o,y_o) does not satisfy the new test for some reasonable number N, say 10.

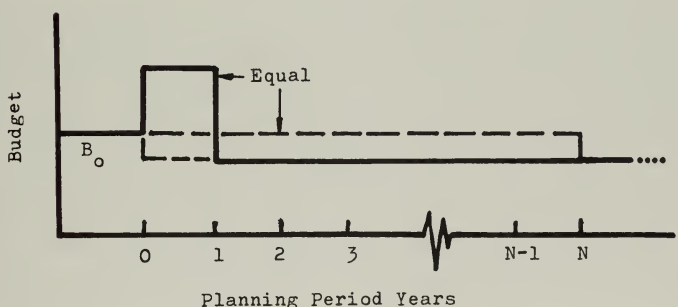
One interpretation of this example is that the fixed stock of budget dollars NB_o could be used to sustain the existing force (x_o,y_o) over the N-year planning period, or the force can be restructured to provide greater effectiveness for the same total program cost.

The time-stream of annual budgets, should a change be made from (x_o,y_o) to the force with maximum φ subject to a program budget NB_o, would be as shown in Figure 6b. The new force composition on the N-year budget constraint cannot be financed in the first year with the budget B_o; only force structures on the $N=1$ locus can do this. The new force composition can be financed by bringing forward future savings in operating costs over N-years to subsidize the investment.

We will now look at the same circumstances from the viewpoint of savings and investment and reach the same result.



6(a)



6(b)

FIGURE 6

Maximum Effectiveness on an N-Year Equal Budget

It may help to refer to Figure 7.

The annual operating cost of the existing force is B_o. Any force on the budget line NB_o will have the same operating cost. Any force included within the region toward the origin will have lower operating costs. We can delineate constant-budget lines for operating costs in this region and label them at their budget level, less than B_o, or label them at increasing annual savings. Similarly we can sketch in constant investment budget lines for additional investment in components x, or y, or $x+y$, above the existing level (x_o, y_o).

The annual savings S over an N-year program planning period will be NS dollar savings. The present value of these annual savings over an infinite time would be S/r, where r is an appropriate interest rate. The annual savings, S, or the present value of the stream of savings depends on which force structure is selected. So does the investment I. What combinations of force components (x,y) will equate the discounted stream of savings and the investment?

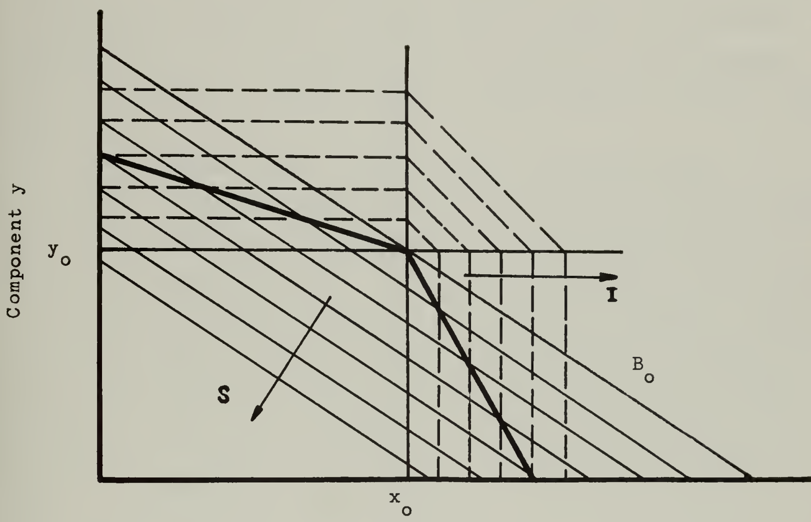
<u>Theorem 4</u>. The locus of forces (x,y) which satisfy the requirement that the sum of constant annual operating savings, discounted at annual rate r, equals the first year investment cost, coincides with the N-year planning period budget constraint, where $N = 1/r$.

The N-year savings NS obviously equal the present value of the stream of savings S/r when $N = 1/r$, and the force combinations for which this occurs will be on the N-year level-budget constraint. The second criterion now examines whether any of these force combinations have higher effectiveness.

<u>Criterion 2</u>: Maximum effectiveness for an N-year level-budget planning period, so that an infinite stream of future savings at interest rate $r = 1/N$, or in a period N years, does not amortize an investment for modification to a more effective force.

Failing this test means that:

(1) If the prevailing judgment of the interest rate is not more than $r = 1/N$, a current investment decision should be considered to modity the force, or



S: Increased Savings.

I: Increased Investment.

▬: Present Value (Savings) = Investment.

FIGURE 7

Present Value of Savings Equals Investment

(2) If the planning period for this force is N years, then an investment to modify the force will be repaid in savings of operating costs in N years.

This criterion is not presented as an optimal policy but as a test of the cost-effectiveness or efficiency of a force structure. In this general analysis, nothing arises to support a particular numerical value for the planning period N-years, or the interest rate r. There does develop as above, good reason to associate the two.

The interest rate r may be used in different ways.

(1) A numerical value from the analysis may be compared with an estimated interest rate from opportunity costs or borrowing costs. It should be cautioned that the benefit in military effectiveness has not been assigned a dollar value. This internally-indicated rate is a measure of how rapidly future savings (not income) offset investment, on the assumption that the savings have continued. If the force structure is not altered, the higher operating costs consume the budget, and higher effectiveness has been foregone.

(2) The indicated rate associated with one project might be compared with that in another project. A high indicated rate for one project means the future savings rapidly offset the investment in that project.

Our future use of this information is that we define an efficient force as one which satisfies at least the N-year test, where N and r have been established at proper values.

CHAPTER IV

EFFICIENT PATH SOLUTIONS

The material in this chapter is presented by introducing two examples. The method for solution is different, but the result for each is an efficient expansion path, $[E(\underline{x});\varphi;B]$. Example 1 is typical of force structures with linear isoquants and is solved by a sequence of linear programs. Example 2 is typical of an effectiveness function with convex isoquants and is solved by a method appropriate to the class of convex functions.

A. Example 1. Linear Effectiveness Isoquants.

The first example is easily visualized on a graphical solution. Assume existing forces (x_o,y_o) with an effectiveness function having linear isoquants, $\varphi = ax + by$, and parametric budget constraints as introduced in Chapter II-C and shown in Figure 3. The problem is: find efficient component pairs $E(x,y)$ which provide maximum effectiveness φ subject to the parametric budget constraint B. The solution is the expansion path $[E(x,y);\varphi;B]$ shown in Figure 8, which is found graphically or by linear programming. The effectiveness function does not have to be linear as long as it is a member of the family given in Chapter II, Section E, which have linear isoquants.

Each point on the expansion path represents an efficient pair $E(x,y)$ in the sense of criteria 1 and 2 of Chapter III. The slope of the budget constraints in quadrants I, II, IV depends on the duration of the planning period, N, and the slopes satisfy the inequalifites repeated here from Chapter II.

$$\mathrm{II} < \mathrm{I} < \mathrm{IV}$$

That is, for all N,

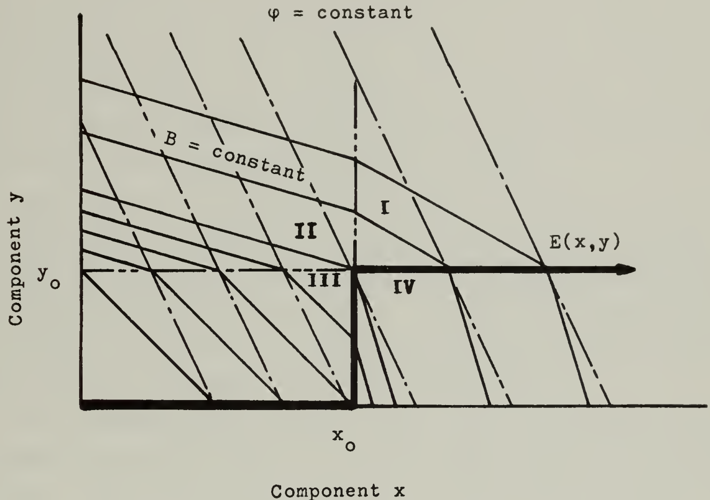


FIGURE 8

Solution for Example One

$$\frac{NO_x}{I_y + NO_y} < \frac{I_x + NO_x}{I_y + NO_y} < \frac{I_x + NO_x}{NO_y} .$$

It can be shown that the slopes in quadrants II and IV are monotonic in N. In quadrant II, the slope increases for increasing N; in quadrant IV, the slope decreases for increasing N. This means that the solution valid for a given N-year period (a given interest rate), is valid for any lesser N-period (higher interest rate). The converse is not true; increasing the planning period, or lowering the interest rate, may completely upset the solution.

The solution in quadrant III satisfies criteria 1 and 2 in the following sense. The slope of the budget constraint in quadrant III is NO_x/NO_y, the ratio of operating costs for existing forces, and is independent of the planning period N. The solution selects the most cost-effective component and this selection satisfies any criteria on interest rate or planning period, in this quadrant.

The mathematically-possible case of the indifferent solution, when budget and effectiveness slopes are equal, must be interpreted with care. The indifferent solution is possible but unlikely in both quadrants I and III because the budget-constraint slopes can be equal in these quadrants. If the indifferent solution occurs in both quadrants, there is no dominant choice for expansion or contraction. If the indifferent solution occurs in only one quadrant of four, then care is recommended to refer to the changes in the slope of the budget constraint in adjacent quadrants and due to a change in planning period. Arbitrary choices within an indifferent selection may well have different operating costs.

In this particular case the existing force (x_o,y_o) is an efficient force, but this in general is not assured. An increase in the planning period or a decrease in the investment cost of additional x units could make x-components dominant over the entire range of φ,B. The expansion path would not go through (x_o,y_o). The expansion path is still the set of forces which are efficient on criteria 1 and 2, but we want to be specific now about the "inefficiency" of the existing force. The situation would be one of the two illustrated in Figure 9. Choose an x investment cost decrease

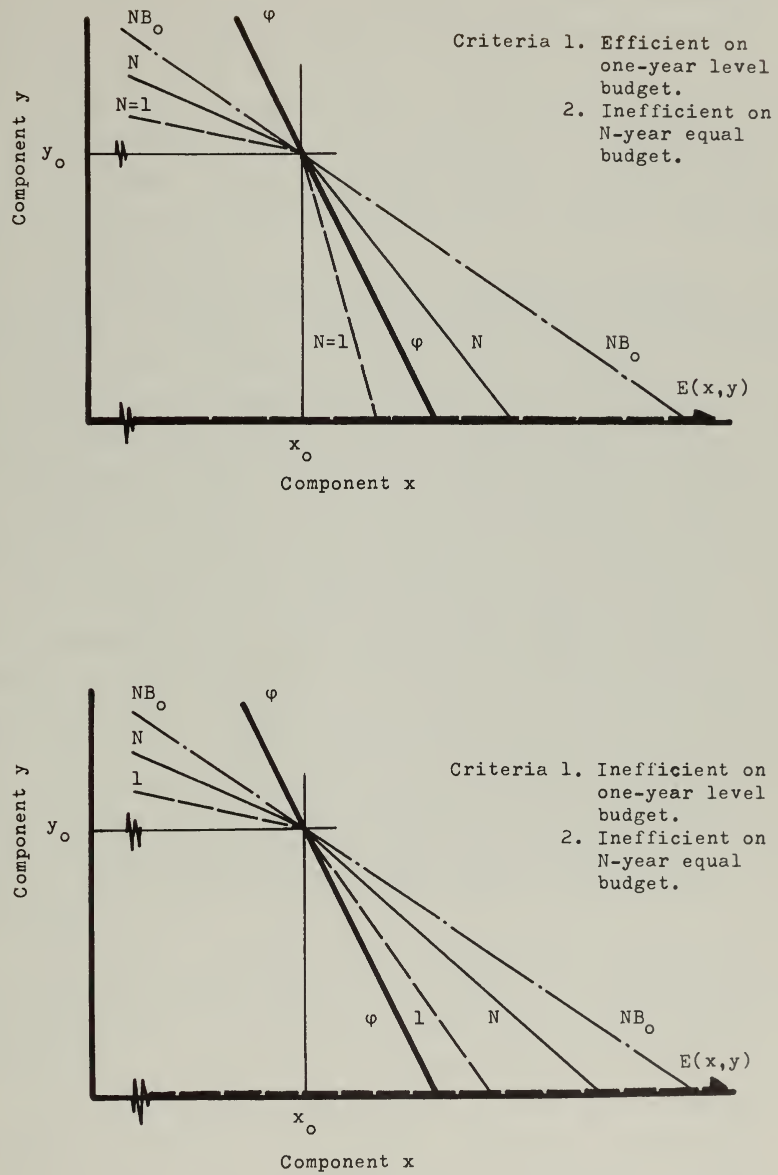


FIGURE 9

Efficient and Inefficient Force Structures

which would decrease the slope of the N-year level budget constraint in quadrant IV to less than the slope of the effectiveness isoquant, and for Figure 9b, further reduce the x investment cost.

In Figure 9a, the existing force is efficient on a one-year basis, and if investment capital is not available, the recommendation may be not to change. On an N-year basis, however, a conversion this year to all-x force will increase effectiveness and decrease operating costs in all future years.

It is interesting to note that once the force is converted for any reason to a lesser y component, there will be no indication in the cost-effective analysis to resupply the y component.

It should also be noted that, in Figure 9a, the potential exists to increase effectiveness, after a temporary loss, with a continuing constant budget B_o. The force can be converted to all-x on the one-year constraint with some loss in effectiveness. The operating costs of the new force are lower, so a continuing constant budget B_o would permit additional x-procurement in succeeding years. In principal, the intersection at NB_o could be reached but at a diminishing rate. Such a policy would require a value judgment on the time history of effectiveness φ which we particularly avoid in this study because the first step is negative. This approach still could be used to explore the possibilities, of course.

In Figure 9b, the existing force is inefficient on both one- and N-year bases. Here it would seem folly not to do something: at least switch to greater effectiveness on the one-year level budget and terminate with lower operating costs as well as higher effectiveness.

In all cases, all points on the expansion path $E(x,y)$ are efficient on both criteria, and any force formed by the modification of an existing force to a new efficient pair $E(x_1,y_1)$ will remain on the path when a revised path is calculated for the new force.

The problem already solved serves as a model for the case of exhaustion of alternatives. If (x_o,y_o) represents the saturation level of one type of (x,y) and substitutes are available at higher cost, the solution is formally identical.

The expansion path for this example is displayed on the x-y plane, but the $E(x,y)$ are associated with the effectiveness φ and budget level B that go with each point. The total solution is the display of $[E(x,y);\varphi;B]$. From the φ,B the analyst can plot total cost of effectiveness and the marginal cost of effectiveness. In this case, the effectiveness increases linearly with budget level, but at different (decreasing) rates for each branch of the path. The marginal cost of effectiveness is constant over each interval and, typically, increases with increased effectiveness.

B. Example 1 with Sequential Decisions: Multi-Year Procurement.

The preceding problem solution provides the efficient path $[E(x,y);\varphi;B]$ for an N-year planning period whose budget program and effectiveness may be projected in chart-form as shown here.

Planning Year	0	1	2	3	...	N	
x-force	x_o	x_1	x_1	x_1	...	x_1	
y-force	y_o	y_1	y_1	y_1	...	y_1	
Effectiveness	φ_o	φ_1	φ_1	φ_1	...	φ_1	
Budget	B_o	B_1	B_2	B_2	...	B_2;	Total B^+

+: Optimal level selected by decision-maker

The budget stream assumes the investment is made in the first year. If this is a large expansion, or if manufacturing schedules simply will not support a one-year expansion leap, should the solution change? If there are sequential decisions about increments of effectiveness or current budget, should the path change?

The efficient path solution does not change if the objective is to maximize the total effectiveness over all the N-years of a program expanding an efficient force. The problem is to consider the program:

Planning Year	0	1	2	3	...	N	
x-force	x_o	x_1	x_2	x_3	...	x_n	
y-force	y_o	y_1	y_2	y_3	...	y_n	
Effectiveness	φ_o	φ_1	φ_2	φ_3	...	φ_n	
Budget	B_o	B_1	B_2	B_3	...	B_n;	Total B^+

The objective is to find $E(x,y)$ which will maximize the total effectiveness $\varphi = \varphi_1+\varphi_2+\ldots+\varphi_N$ subject to a total budget B. We only consider expansion of an efficient force and require that the force each year to be at least equal to the force existing the previous year so that the program budget B must be at least NB_o. When the budget B is greater than NB_o, expansion is possible and the solution is always that all expansion take place in the first year. The sensitive assumption here is that the total budget B may be distributed any way in the period consistent with the force, so there is a big first-year investment. If the budget increment in the first year is larger than the decision-maker allows, the reduction in the first year has the effect of reducing the total program budget. In succeeding years, new increments of investment funds would still be spent in the first year they become available.

The expansion path for sequential decisions based on an existing force not efficient on both one and N-year levels is sensitive to assumptions. The problem may be visualized by comparison of the expansion paths in Figures 8, 9a and 9b. Figure 8 is the simplest because the force is efficient on a one-year criteria, E_1, and is efficient on the N-year criteria, E_N, illustrated. The paths for one year and N-year optimality are confluent and all points are both E_1 and E_N.

In Figure 9a, the E_N path does not include the existing (x_o,y_o). The existing (x_o,y_o) is E_1, however, so a path based on E_1 would branch up to (x_o,y_o) and "look like" Figure 8. The points on the E_1 path would be E_1 but not E_N. The points on the E_N path are both E_1 and E_N. The importance of this difference is that if the sequential budget decision is at or near the one-year level, the analysis shows a first-year drop in effectiveness to get the N-year path. The alternative to the drop in effectiveness is to follow the E_1 path and perpetuate a force that is not E_N. The value-question of a drop in current effectiveness against long-run economy is not amenable to this analysis until a decision is given on the time-phased value of effectiveness. If the expansion is large, the larger budget will finesse the problem.

In Figure 9b, the existing force is neither E_1 nor E_N.

The expansion path on either basis bypasses (x_o, y_o). It would seem not prudent under this analysis to perpetuate y_o in any decision based on these facts. This happy situation is unfortunately due in part to the linearity of the effectiveness isoquants yielding a dominant choice. Here, it is clear what to recommend. The similar problem with an inefficient force on E_1 and E_N is not self-evident when the effectiveness has convex isoquants. Then once again, time-phasing of effectiveness versus the sequence of budget levels must be weighed by the decision-maker. Once again no attempt is made here to make a judgment on the time-stream of effectiveness. If time-phased requirements are given, these are simple time milestones along the expansion path. No "weighting function" for relative desirability of time-phased effectiveness can be generated by the analyst, since effectiveness and budget levels are judgments left to the decision-maker.

In summary, sequential decisions do not change the preferred expansion path, given that the criteria for efficient combinations do not change. It is, however, not possible to modify an existing force that is not E_1 and E_N to the efficient path without a temporary loss of effectiveness if the budget is too low.

C. Example 1 with Multiple Alternatives.

The preceding problems were given in terms of two alternatives simply because the two-dimentional problem is more easily visualized. There is no basic difficulty in extending this approach to multiple alternatives, since there is no logical difference in extending the problem to an n-dimensional linear program. In this case, the isoquants and budget constraints are hyperplanes. As a practical matter, the expansion path would always start at the origin, since zero force, effectiveness and budget are consistent, so the origin will always be a basic feasible solution, and the path builds from there.

D. Example 2. Convex Effectiveness Isoquants.

This example is introduced as distinct from the Example 1 in the techniques for solution and a discussion of force ratios.

The techniques suggested here are already part of mathematical

literature. No special conclusions depend upon the technique chosen. The point is, as before, to find $[E(\underline{x});\varphi;B]$.

The efficient path may recommend to expand or contract in a changing ratio, or a set of discrete ratios depending on the level authorized. As a practical matter in the case of linear programming, one expects a dominant alternative at each junction, and the expansion path is parallel to an axis. It is "either/or." With convex isoquants, the dominant alternative is not usual. This continuous range of potential force combinations calls for a further review of the criteria for "efficient" compositions.

We refer now to Chapter II, Part II, <u>Effectiveness Functions</u>, to admit any continuous, differentiable, function whose isoquants are convex to the origin. This is such a gigantic family of functions that we would be distracted by an enumeration of all the computational techniques one might need to conquer all. In Section F of Chapter II we were more specific, and we can list some of the forms that are readily admitted to see what a few "look like."

1. Quadratic forms: $\varphi = k'\underline{x} + \underline{x}'A\underline{x}$.

2. Product forms: $\varphi = \prod^{n} x_1^{a_i} = x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots x_n^{a_n}$

3. Homogeneous forms: $\varphi(t\underline{x}) = t^n\varphi(\underline{x})$

4. Constant-elasticity forms:

$$\varphi = F[((1-\delta)a^{1/e} + \delta b^{1/e})^e] \ , \ \eta \neq 1$$

$$F(a^{1-\delta}b^{\delta}) \ , \ \eta = 1 \ .$$

This listing is made (and probably can be expanded) to illustrate application of different techniques and for different interpretations. The quadratic forms invite treatment by a particular array of the techniques of nonlinear programming called quadratic programming. The product forms have the special characteristic that the isoquants can not intersect any axis $x_j=0$ and corner solutions at an axis do not occur, so standard differential calculus can be used when interpreted as will be shown. Homogeneous forms which also have constant total elasticity have simplified functional form.

"Constant-elasticity forms" takes care of the many functions which satisfy Theorems 2 and 3 and have piecewise-linear expansion path solutions.

Example 2 is a Product Form, from Chapter II-F, also used in Chapter I,

$$\emptyset = \left(\frac{L_A}{L_B}\right)^a \left(\frac{M_A}{M_B}\right)^b ,$$

where $\emptyset$ is the effectiveness of Force-A components L_A, M_A against Force-B fixed components L_B, M_B. The form to be studied with a=0.8, b=0.4, is

$$\varphi = x^{0.4}\, y^{0.8} .$$

In order to show the dependence of the expansion path on existing forces, we take two cases of assumed existing forces,

$$F_1 = (0.10,\ 0.50) ,$$

$$F_2 = (0.325,\ 0.275) .$$

Take investment and annual operating costs per unit as given in the following table. For criterion 2, we specify the consistent values of discount rate and planning period, $r = 1/N$, to be $0.10 = 1/10$.

Costs:	Investment	Annual Operation	N-Year Operation	I + NO
Input x:	100.	10.	100.	200.
Input y:	50.	10.	100.	150.

The given information is presented in Figure 10, which shows some equal-effectiveness isoquants and expresses the cost information as parametric 10-year budget constraints based on force (0.10, 0.50). A similar budget-cost field would exist in four quadrants around the other assumed force. The problem is to find the expansion path $E(x,y)$ which will maximize φ subject to the budget constraint.

The budget constraints are piecewise linear and the effectiveness contours are convex to the origin. We expect to find tangent or corner solutions but not on an x,y axis. Although more formal attack can be planned, we simplify the solution by making some initial

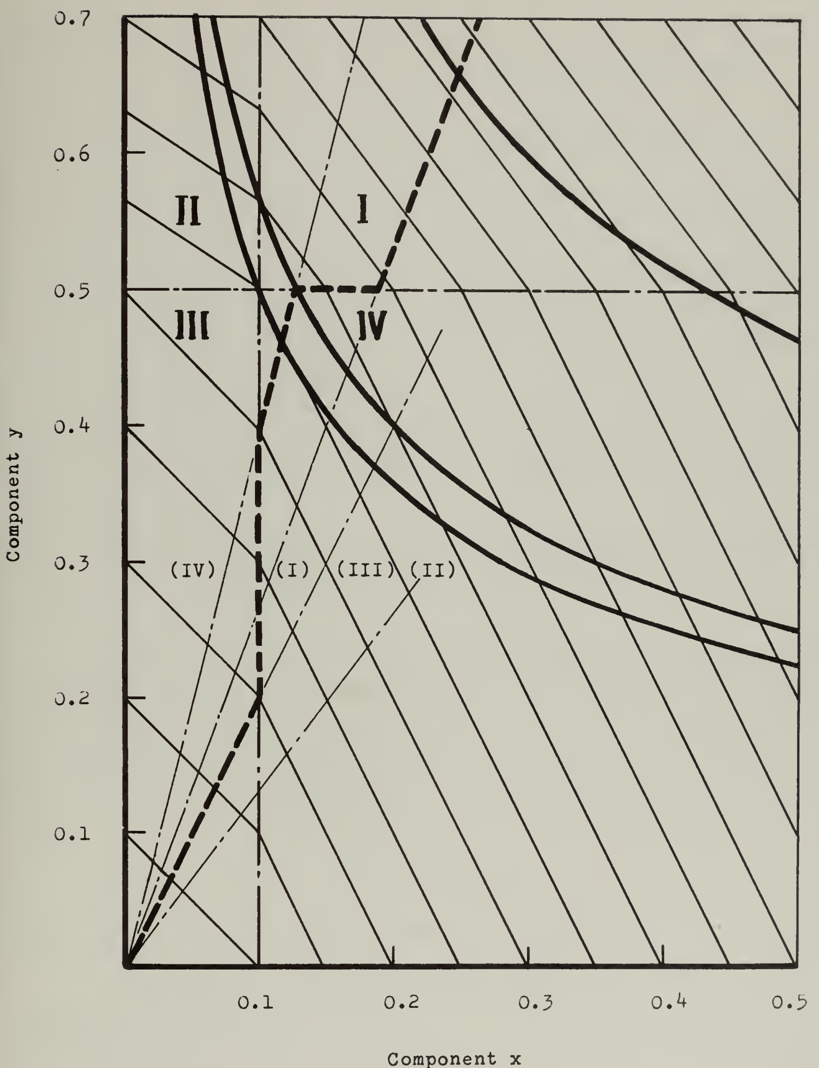


FIGURE 10

Introduction to Example Two

observations.

(1) There are four quadrants around the existing force point taken as an origin.

(2) Within each quadrant, the budget constraint is linear with constant slope.

(3) If an efficient solution exists within each quadrant, it will be a tangent solution.

(4) The locus of efficient solutions within each quadrant, if they exist, will be straight line rays from the origin, (Theorem 3). These may be components of the expansion path.

(5) Each ray is valid only in its budget quadrant.

(6) The four rays are independent of the existing force; their region of validity is dependent on the existing force.

(7) The transition solutions between quadrants are collinear with the quadrant axes. They correspond to corner solutions.

These observations are sufficient to permit the synthesis of the expansion paths as shown in Figure 10 for force F_1 and Figure 11 for both F_1 and F_2. The four rays are shown and are given the number of their quadrant. The slopes of the rays are in the same relation as the slopes of the budget constraints, by name and slope,

$$\mathrm{II} < \mathrm{III} < \mathrm{IV}$$
$$\mathrm{II} < \mathrm{I} < \mathrm{IV} \quad ,$$

and in this numerical example $\mathrm{I} > \mathrm{III}$ for all N, which is 10 here.

We will follow the path in Figure 10. The initial ray in quadrant III corresponds to operation of existing force elements in the ratio of two to one. This ratio is much less than the existing ratio of five to one, so the existing x-units are exhausted first. After the existing x are all in operation, the cheapest way to buy effectiveness is to operate additional existing y-forces, until the ratio is four to one. At the ratio four to one, the ray valid for quadrant IV is encountered. That is, the marginal product-cost ratio of the remaining y-forces alone is less than that for the

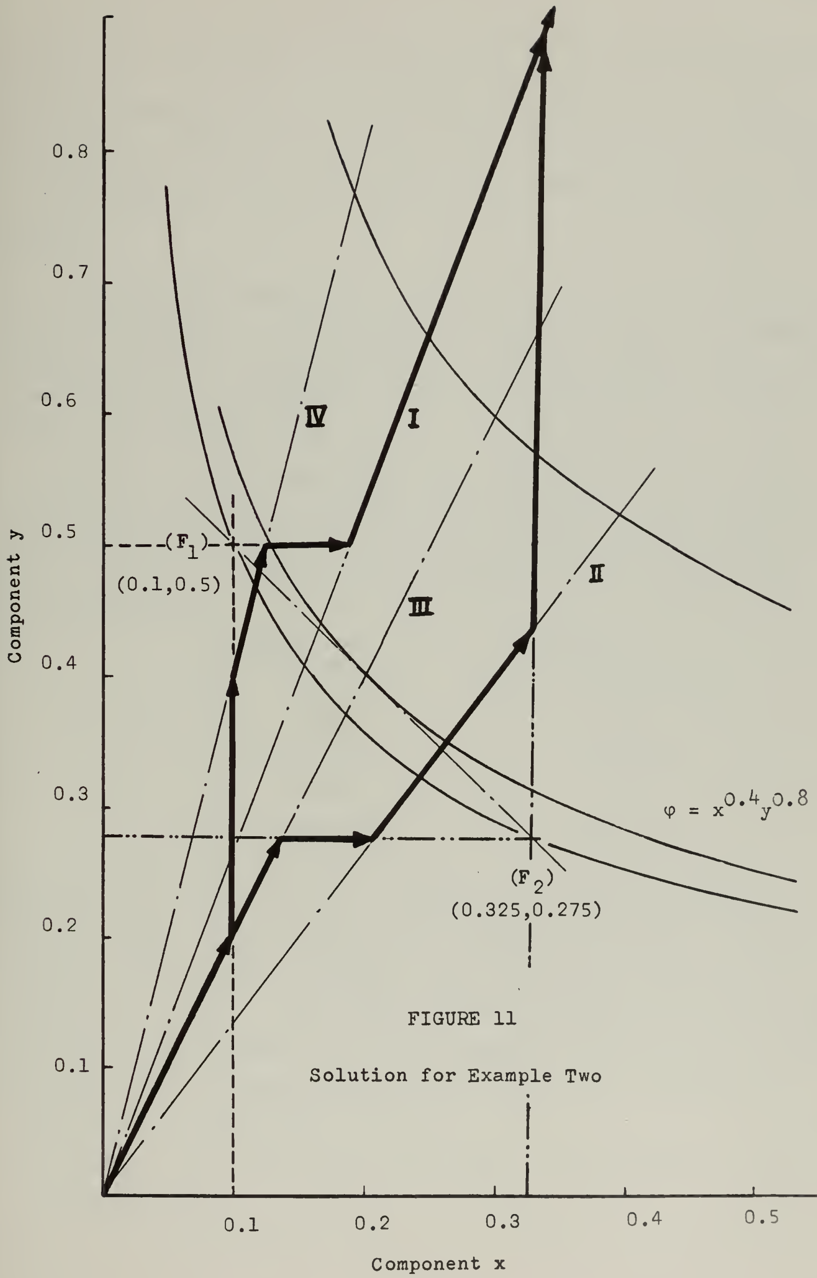


FIGURE 11

Solution for Example Two

combination of existing y plus the purchase of new x-units in the ratio four to one. This ratio continues to be the most effective feasible combination until the existing y-forces are all in operation. Then only x-forces are added until the force ratio is eight to three, corresponding to the ray for quadrant I. Here it is most efficient to purchase both x and y, not either alone, for any further expansion.

The product-type effectiveness function is nicely displayed in log-log coordinates. The isoquants are parallel lines with negative slope, and the four rays are parallel lines with positive slope. The expansion path is formed from a sequence of rays and axial transitions, as in Figure 12.

The identical expansion solution applies to the following examples of exhaustion of alternatives. Assume two cases with costs and availabilities as shown in the table. Units have been normalized so that x_1 is a unit substitute for x_2 and (y_1, y_2) are unit substitutes.

	Case 1		Case 2	
	Price	Availability	Price	Availability
x_1	1.0	0.100	1.0	0.325
x_2	2.0	Unlimited	2.0	Unlimited
y_1	1.0	0.500	1.0	0.275
y_2	1.5	Unlimited	1.5	Unlimited

The role of "existing forces" (x_o, y_o) is now taken by the "exhaustion crossover" (x_o, y_o) and the budget constraint is piecewise linear in four quadrants around the exhaustion crossover. In each quadrant, the consumer rationally takes the cheapest available combination of (x_i, y_j). The budget slopes are identical to the slopes of the preceding example. For each quadrant, the same ray is found as before, and the path synthesized as before. The solution is illustrated in Figures 10, 11 and 12.

E. Expansion and Contraction Contrasted: Irreversibility of the Efficient Path.

If the existing force F_1 were expanded to a point on the

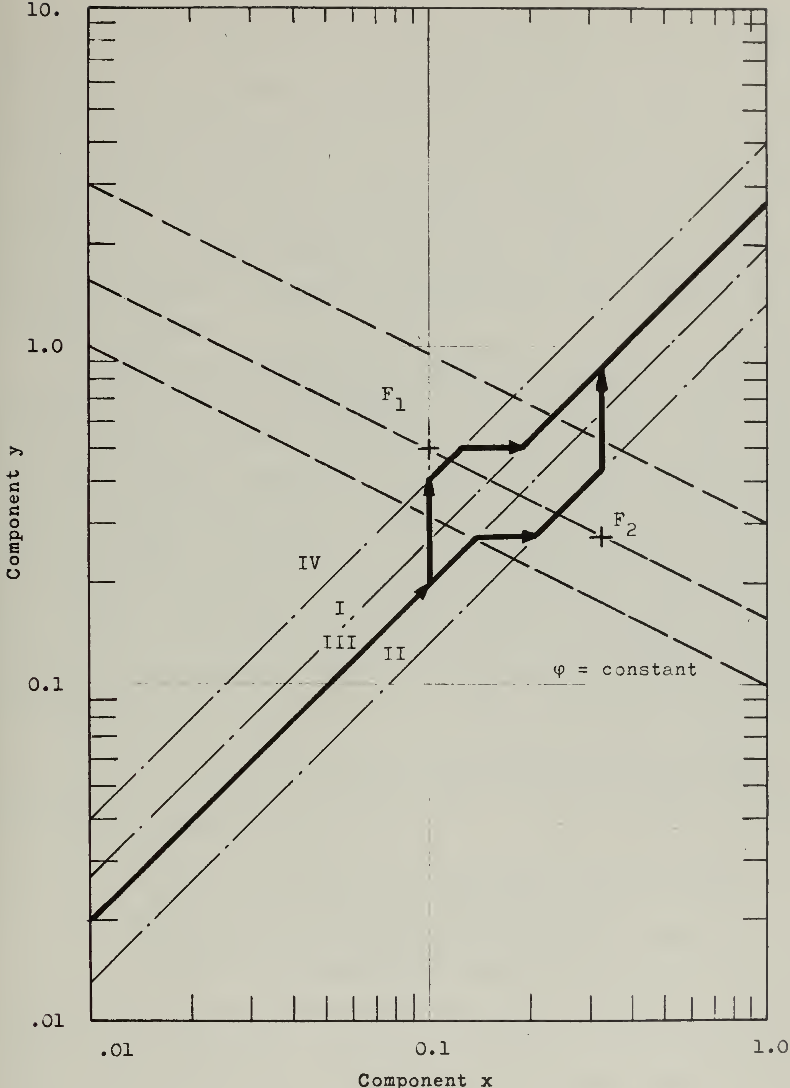


FIGURE 12

Log-Log Graphics for Solution Two

expansion path, say (0.225, 0.60), the budget constraints would shift as shown in Figure 13 in contrast to those of Figure 10. The four rays do not change, but the boundaries of the regions of validity have shifted. The new path is also shown in Figure 13. The new force combination remains on the new path. Observe, however, that a contraction of this force would never return to the original F_1 composition. In a contraction, this force should first retire y units until the most effective force ratio for operating costs is reached at (0.225, 0.450), and maintain this fixed ratio in any further reduction.

The expansion path clearly depends on the initial assets, if there are any, and if the investment cost is not zero. When there are no existing forces, we prefer to say that (x_o, y_o) are (0,0) and only quadrant I applies. The problem is still best illuminated by the expansion path, but the solution is not more difficult than for the example of Chapter I. If the investment costs are zero (or negligible relative to annual operating costs), then independent of (x_o, y_o) the slopes of all rays collapse to the one for quadrant III. Again, the simple example of Chapter I is recovered. In the absence of these special cases, the expansion path always depends on the existing force.

If the force is modified, the expansion path is modified. Let us say that at least the first step is to an efficient force on an efficient expansion path.

(1) If the force sequentially expands, the contraction branch is sequentially modified in part.

(2) If the force sequentially contracts, the expansion branch is sequentially modified in part.

If these force changes are "mild" expansions or "mild" contractions, the efficient expansion or contraction will reflect the past history of the force composition. If the changes are large, however, all trace of the past history is removed. If a force contracts, the loss of some component may not be replaced in a subsequent expansion. The concept of an efficient force thus shows irreversibility. What <u>was</u> efficient may not be efficient after a change.

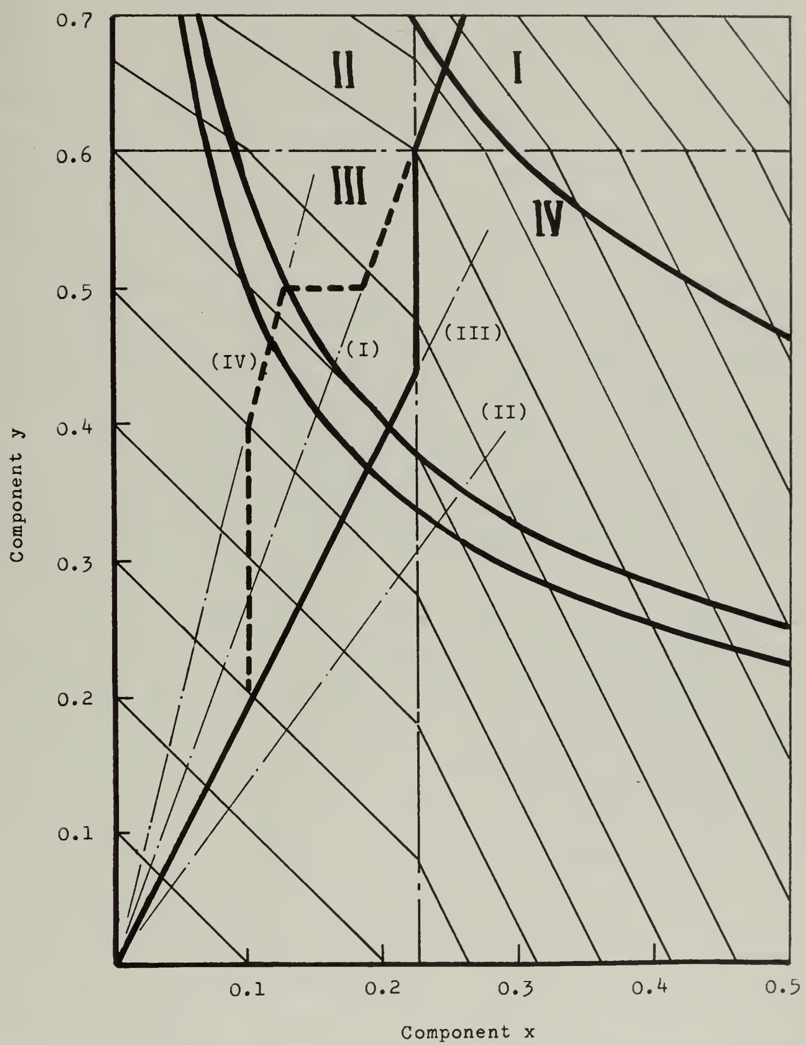


FIGURE 13

Example Two After Expansion

The irreversibility is illustrated by modifying the force structure F_1 of Example 2 as shown in Figure 14. The initial force F_1 may be expanded "mildly" to point C, which retains the initial y_o and only adds x-units, or F_1 may expand more vigorously to point D, on the quadrant-I ray and no longer reflect y_o. The contraction path from C is C-F-A. In an expansion from A, the path would now be A-B-D, never recovering either F_1 or C. Similarly the contraction from D is D-G-F-A.

Contractions of force F_1 are shown in Figure 15. A "mild" contraction here could be relatively large in y, down to point B, before any x-units were retired, as they are at point A. Typical expansions after contraction would be C-F-D or A-G-F-D.

Note that after an extreme demobilization of y-units from F_1 to A (in either figure), a subsequent expansion almost immediately begins adding x-units as well as replacing y-units. Such tracings of the expansion paths should help to make clear that what was efficient may no longer be.

One more example, in Figure 14. The Force F_1 might be expanded to say, point D. If the point D level of operating costs and effectiveness are too great for the need, a contraction to point G might be in order. Notice that x and y forces have the same operating costs per unit, but the x investment costs are greater. The higher sunk costs in x are protected and only y-forces would be retired at first, down to G. At G, the operating budget happens to be the same as the old F_1, but the effectiveness is higher. A contraction to "the good old F_1" would have retained more of the original y-units, and the operating budget would have been cut back just as far — but the effectiveness is definitely less. The efficient paths have some irreversibility. We can also see that further contraction from G should be in the force ratio of two to one, and not one to one just because operating costs of each unit are equal.

F. Example 2 with Sequential Decisions or Multiple Alternatives.

To consider sequential decisions, it is important to recognize the difference caused by an existing force that is, or is not, efficient. If the existing force, the starting point of a sequence, is

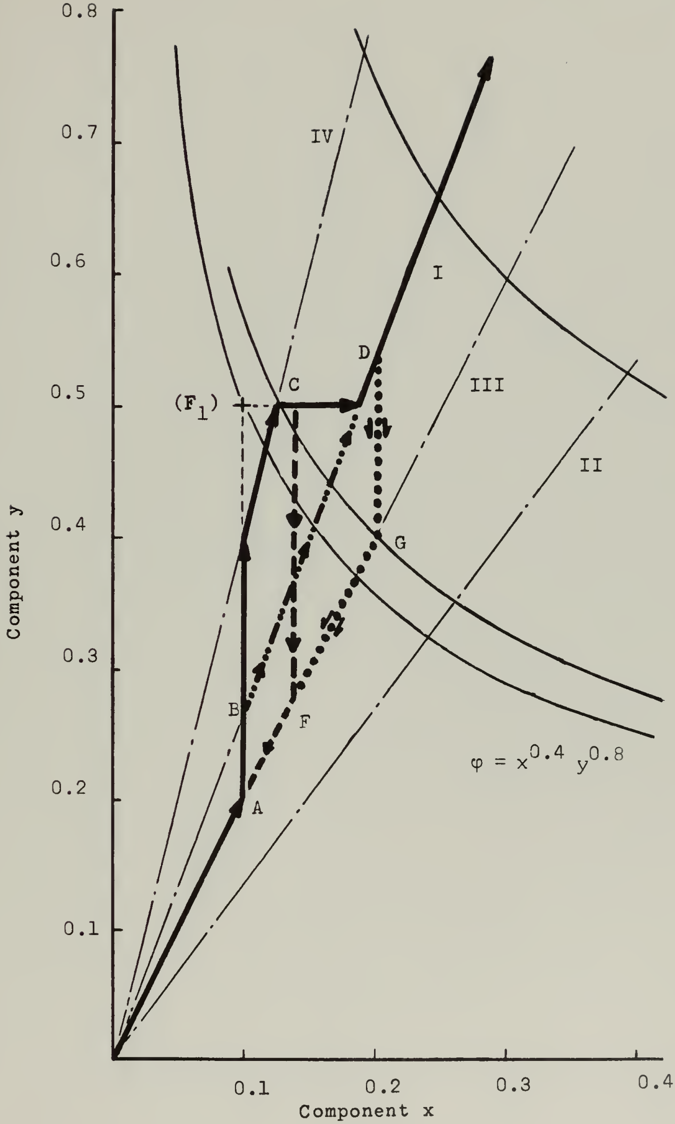


FIGURE 14

Sequential Expansion of Force F_1

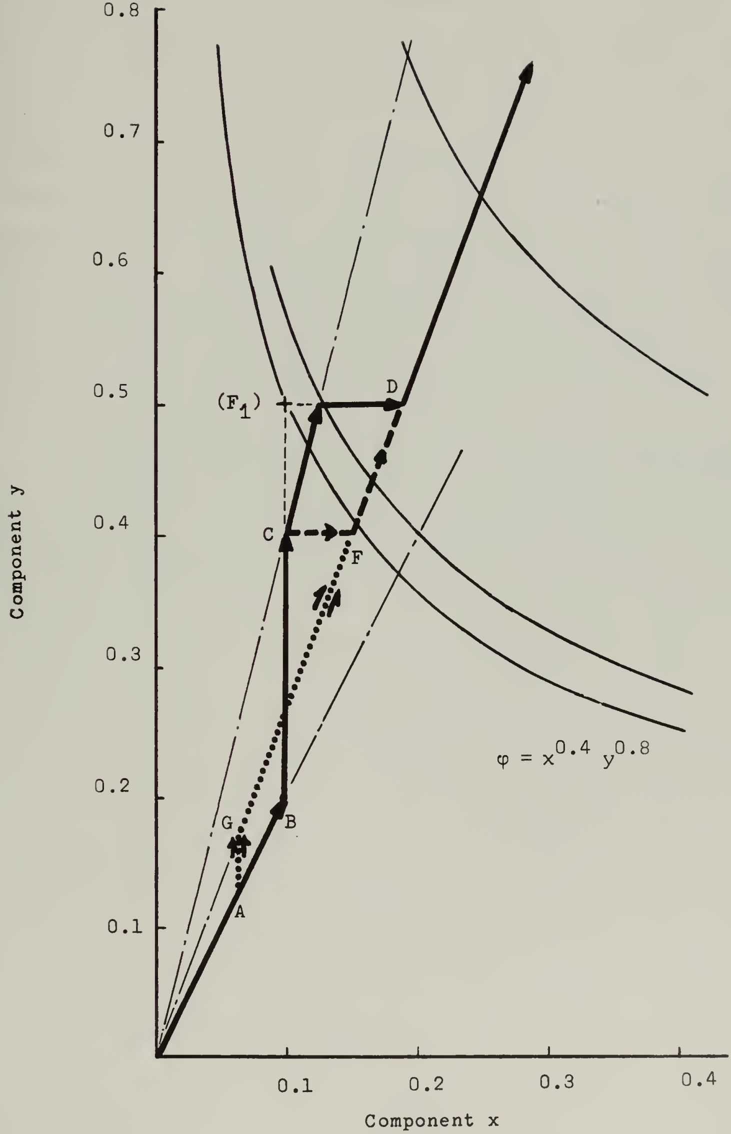


FIGURE 15

Sequential Contraction of Force F_1

efficient, then it is on the expansion path. The end point of the force change must also be on the expansion path to be efficient. An intermediate point in the sequence not on the path would not be efficient. Selecting an intermediate inefficient point off the path would imply some judgment of the relative preference of a time stream of effectiveness against cost. Although we have found a natural relation between the planning period and the interest rate appropriate for the analyst to use for cost discounting, the time-weighting of effectiveness requirements must be a stipulation to the analyst. For the same reason, the modification of an inefficient force, if not changed enough to become an efficient force on the path, requires stipulation of the time-weighting of effectiveness. The situation should not be surprising to readers in economics who recognize the parallel with utility theory.

Multiple alternatives do not change the general approach, as long as we maintain the requirement for convexity of the isoquants. Then, thinking of the rays from the origin as a convex cone, one can synthesize a multi-dimensional path from the appropriate ray segment with axial transitions.

CHAPTER V

CONCLUSIONS

A. Expansion or Contraction of An Efficient Force.

Expansion paths have been described as an informative analytic result for studies of expansion or contraction of existing forces. The procedure for finding the path, using an N-year planning period budget constraint, results in efficient forces $E(\underline{x})$ which satisfy criteria for efficiency.

Two criteria are invoked: maximum effectiveness on both a one-year level budget, and an N-year budget of equal total amount. A force, existing or planned, which fails the one-year test is inefficient. The study of modifications to inefficient forces is discussed in the next section. A force which passes both tests is efficient. There are probably many force structures in a given problem that are efficient; they are in a convex cone from the origin. Only some of those efficient forces are accessible on an efficient change from an existing force structure. They are accessible by the expansion path. Some of the efficient forces that are accessible shift with each change in the existing force. This is the modification of the expansion or contraction path with each change in force structure.

Once an existing force is an efficient combination, modification of the force to another efficient combination will retain some of the expansion path. Only the opposite branch of the path is revised after each change. The N-year path, corresponding to an implied interest rate $r = 1/N$, consists of combinations $E(\underline{x})$ which will also satisfy a criterion modified for any $N' < N$; that is, for any higher natural interest rate.

The expansion path is represented by a function of the force components which has been called $E(\underline{x})$, or $E(x,y)$ in two dimensions, such as $y=kx$, or the piecewise linear relations of (y,x)

shown graphically in the figures of Chapter IV. There is no explicit statement of φ or B in $E(\underline{x})$ itself. However, with each vector $\underline{x}$ is associated one and only one φ and B. The efficient solution is a display of the three consistent quantities $[E(\underline{x});\varphi;B]$. At the beginning of the problem of force structure analysis, there must be a budget function $B(\underline{x})$, and there must be an effectiveness function $\varphi(\underline{x})$, and in the beginning these functions of the same variable are independent. For the same budget B, one can have many levels of effectiveness, or one might pay a wide range of budget costs for an effectiveness φ. It is the marvelous property of the function $E(\underline{x})$ to destroy this independence in the most efficient way. The $E(\underline{x})$ relates the force components in such a way that for the budget B we get only one effectiveness φ, the maximum effectiveness available under some criteria. Conversely, for one effectiveness φ, we need only pay the least possible B. Very briefly, $E(\underline{x})$ is simply the function which causes the Jacobian of the effectiveness function and budget function to vanish. Then the total solution is the triumvirate $[E(\underline{x});\varphi;B]$ of which only one is independent. With the three functions, one can deduce any combination such as force composition, total cost, cost of effectiveness, force effectiveness, and marginal cost of effectiveness.

The expansion path is a functional solution to the force structure problem. It is not meant to provide single answers which imply no alternatives. The analyst has no foundation for usurping the prerogative of the decision-maker, nor sufficient information to judge the appropriate level of effectiveness or budget. These can only be determined at a higher level which can weigh political and economic benefit or consequence of this program against the threat, political intentions, and fiscal impact. The analyst can, with the expansion path information, provide a spectrum of efficient alternatives and their consequence for background to the decision.

B. Introduction to Analysis of Alternatives for Modification of an Inefficient Force Structure.

Chapter III presented two criteria for judging efficiency. If a force is judged inefficient on the first, then effectiveness can be improved within the one-year level budget. It is possible,

as illustrated in Figure 4, to modify the force structure on the level budget, improve effectiveness, and reduce the budget in all following years to the lower operating cost. We do not rush ahead with the recommendation, however, because other important alternatives suggest themselves. For example, in Figure 4, one could also increase the x-force and reduce y a little more, on the same level budget, and get the same effectiveness as the original, not increased. But now the future savings are even greater. In fact, a selection of effectiveness levels, not less than the original, are available on the same one-year budget, all with various savings on future operating costs.

Still other avenues are opened by other assumptions. Suppose the level budget were to continue forever, and the force commander were free to develop his structure for greatest efficiency. Movement on the capability plane within the budget constraints can be planned on a sequential basis, and higher levels of effectiveness achieved, in principle. The possibilities offered by continuing the level budget of an inefficient force, for example, could yield an ever-increasing effectiveness but with diminishing returns each year as the initial inefficient force (x_o,y_o) has a sufficient level budget to allow small (and smaller and smaller) procurements each year until the NB_o limit line (Figure 2) is reached.

There are two major difficulties that extend the study of such possibilities for inefficient forces beyond this report.

(1) To judge alternatives will require a comparison of time-phased effectiveness which we have not admitted, and,

(2) the infinite alternatives are very sensitive to the assumptions of the decisions that might be made about time-phased effectiveness or budget levels.

Perhaps some further study will classify the problems among alternatives for modifying inefficient forces by some general principles, and the problem illuminated.

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